A Spectral Method for the Eigenvalue Problem for Elliptic Equations

Kendall Atkinson
Departments of Mathematics & Computer Science
The University of Iowa

Olaf Hansen
Department of Mathematics
California State University San Marcos

September 21, 2009

Abstract

Let Ω be an open, simply connected, and bounded region in $\mathbb{R}^d$, $d \geq 2$, and assume its boundary $\partial\Omega$ is smooth. Consider solving the eigenvalue problem $Lu = \lambda u$ for an elliptic partial differential operator L over Ω with zero values for either Dirichlet or Neumann boundary conditions. We propose, analyze, and illustrate a 'spectral method' for solving numerically such an eigenvalue problem. This is an extension of the methods presented earlier in [5], [6].

1 INTRODUCTION

We consider the numerical solution of the eigenvalue problem

$$Lu(s) \equiv -\sum_{k,\ell=1}^{d} \frac{\partial}{\partial s_k}\left(a_{k,\ell}(s)\frac{\partial u(s)}{\partial s_\ell}\right) + \gamma(s)u(s) = \lambda u(s), \qquad s \in \Omega \subseteq \mathbb{R}^d \quad (1)$$

with the Dirichlet boundary condition

$$u(s) \equiv 0, \qquad s \in \partial\Omega. \quad (2)$$

Assume $d \geq 2$. Let Ω be an open, simply–connected, and bounded region in $\mathbb{R}^d$, and assume that its boundary $\partial\Omega$ is smooth and sufficiently differentiable. Similarly, assume the functions $\gamma(s)$ and $a_{i,j}(s)$, $1 \leq i, j \leq d$, are several times continuously differentiable over $\overline{\Omega}$. As usual, assume the matrix $A(s) = [a_{i,j}(s)]$ is symmetric and satisfies the strong ellipticity condition,

$$\xi^{\mathrm{T}} A(s)\xi \geq c_0 \xi^{\mathrm{T}}\xi, \qquad s \in \overline{\Omega}, \quad \xi \in \mathbb{R}^d \quad (3)$$

with $c_0 > 0$. For convenience and without loss of generality, we assume $\gamma(s) \geq 0$, $s \in \Omega$; for otherwise, we can add a multiple of $u(s)$ to both sides of (1), shifting the eigenvalues by a known constant.

In the earlier papers [5] and [6] we introduced a spectral method for the numerical solution of elliptic problems over Ω with Dirichlet and Neumann boundary conditions, respectively. In the present work, this spectral method is extended to the numerical solution of the eigenvalue problem for (1)–(2), and in a later section it is also extended to the Neumann problem

$$\begin{aligned} -\Delta u(s) &= \lambda u(s), \qquad s \in \Omega \\ \frac{\partial u}{\partial n} &= 0, \qquad s \in \partial\Omega. \end{aligned}$$

2 The Dirichlet problem

Our spectral method is based on polynomial approximation on the unit ball B_d in $\mathbb{R}^d$. To transform a problem defined on Ω to an equivalent problem defined on B_d, we review some ideas from [5] and [6], modifying them as appropriate for this paper.

Assume the existence of a function

$$\Phi : \overline{B}_d \underset{onto}{\overset{1-1}{\longrightarrow}} \overline{\Omega} \tag{4}$$

with Φ a twice–differentiable mapping, and let $\Psi = \Phi^{-1} : \overline{\Omega} \underset{onto}{\overset{1-1}{\longrightarrow}} \overline{B}_d$. For $v \in L^2(\Omega)$, let

$$\widetilde{v}(x) = v(\Phi(x)), \qquad x \in \overline{B}_d \subseteq \mathbb{R}^d \tag{5}$$

and conversely,

$$v(s) = \widetilde{v}(\Psi(s)), \qquad s \in \overline{\Omega} \subseteq \mathbb{R}^d. \tag{6}$$

Assuming $v \in H^1(\Omega)$, we can show

$$\nabla_x \widetilde{v}(x) = J(x)^{\mathrm{T}} \nabla_s v(s), \qquad s = \Phi(x)$$

with $J(x)$ the Jacobian matrix for Φ over the unit ball B_d,

$$J(x) \equiv (D\Phi)(x) = \left[\frac{\partial \varphi_i(x)}{\partial x_j}\right]_{i,j=1}^{d}, \qquad x \in \overline{B}_d. \tag{7}$$

To use our method for problems over a region Ω, it is necessary to know explicitly the functions Φ and J. We assume

$$\det J(x) \neq 0, \qquad x \in \overline{B}_d. \tag{8}$$

Similarly,

$$\nabla_s v(s) = K(s)^{\mathrm{T}} \nabla_x \widetilde{v}(x), \qquad x = \Psi(s)$$

with $K(s)$ the Jacobian matrix for Ψ over Ω. By differentiating the identity

$$\Psi\left(\Phi\left(x\right)\right) = x, \qquad x \in \overline{B}_d$$

we obtain

$$K\left(\Phi\left(x\right)\right) = J\left(x\right)^{-1}. \tag{9}$$

Assumptions about the differentiability of $\widetilde{v}\left(x\right)$ can be related back to assumptions on the differentiability of $v(s)$ and $\Phi(x)$.

Lemma 1 *If* $\Phi \in C^k\left(\overline{B}_d\right)$ *and* $v \in C^m\left(\overline{\Omega}\right)$, *then* $\widetilde{v} \in C^q\left(\overline{B}_d\right)$ *with* $q = \min\left\{k, m\right\}$.

Proof. A proof is straightforward using (5). ■

A converse statement can be made as regards $\widetilde{v}$, v, and Ψ in (6).

Consider now the nonhomogeneous problem $Lu = f$,

$$Lu(s) \equiv -\sum_{k,\ell=1}^{d} \frac{\partial}{\partial s_k}\left(a_{k,\ell}(s)\frac{\partial u(s)}{\partial s_\ell}\right) + \gamma(s)u(s) = f(s), \qquad s \in \Omega \subseteq \mathbb{R}^d. \tag{10}$$

Using the transformation (4), it is shown in [5, Thm 2] that (10) is equivalent to

$$\begin{aligned} -\sum_{k,\ell=1}^{d} \frac{\partial}{\partial x_k}\left(\widetilde{a}_{k,\ell}(x)\det\left(J(x)\right)\frac{\partial \widetilde{v}(x)}{\partial x_\ell}\right) + \left[\widetilde{\gamma}(x)\det J(x)\right]\,\widetilde{u}(x) \\ = \widetilde{f}\left(x\right)\det J(x), \qquad x \in B_d \end{aligned} \tag{11}$$

with the matrix $\widetilde{A}\left(x\right) \equiv \left[\widetilde{a}_{i,j}(x)\right]$ given by

$$\widetilde{A}\left(x\right) = J\left(x\right)^{-1} A\left(\Phi\left(x\right)\right) J\left(x\right)^{-\mathrm{T}}. \tag{12}$$

The matrix $\widetilde{A}$ satisfies the analogue of (3), but over B_d. Thus the original eigenvalue problem (1)–(2) can be replaced by

$$\begin{aligned} -\sum_{k,\ell=1}^{d} \frac{\partial}{\partial x_k}\left(\widetilde{a}_{k,\ell}(x)\det\left(J(x)\right)\frac{\partial \widetilde{u}(x)}{\partial x_\ell}\right) + \left[\widetilde{\gamma}(x)\det J(x)\right]\,\widetilde{u}(x) \\ = \lambda\widetilde{u}(x)\det J(x), \qquad x \in B_d \end{aligned} \tag{13}$$

As a consequence of this transformation, we can work with an elliptic problem defined over B_d rather than over the original region Ω.

2.1 The variational framework

To develop our numerical method, we need a variational framework for (10) with the Dirichlet condition $u = 0$ on $\partial\Omega$. As usual, multiply both sides of (10)

by an arbitary $v \in H_0^1(\Omega)$, integrate over Ω, and apply integration by parts. This yields the problem of finding $u \in H_0^1(\Omega)$ such that

$$\mathcal{A}(u,v) = (f,v) \equiv \ell(v), \qquad \text{for all } v \in H_0^1(\Omega) \tag{14}$$

with

$$\mathcal{A}(v,w) = \int_\Omega \left[\sum_{k,\ell=1}^{d} a_{k,\ell}(s) \frac{\partial v(s)}{\partial s_\ell} \frac{\partial w(s)}{\partial s_k} + \gamma(s) v(s) w(s) \right] ds, \quad v, w \in H_0^1(\Omega). \tag{15}$$

The right side of (14) uses the inner product $(\cdot,\cdot)$ of $L^2(\Omega)$. The operators L and $\mathcal{A}$ are related by

$$(Lu, v) = \mathcal{A}(u,v), \qquad u \in H^2(\Omega), \quad v \in H_0^1(\Omega), \tag{16}$$

an identity we use later. The function $\mathcal{A}$ is an inner product and it satisfies

$$|\mathcal{A}(v,w)| \le c_{\mathcal{A}} \|v\|_1 \|w\|_1, \qquad v, w \in H_0^1(\Omega) \tag{17}$$

$$\mathcal{A}(v,v) \ge c_e \|v\|_1^2, \qquad v \in H_0^1(\Omega) \tag{18}$$

for some positive constants $c_{\mathcal{A}}$ and c_e.

Associated with the Dirichlet problem

$$Lu(x) = f(x), \qquad x \in \Omega, \quad f \in L^2(\Omega) \tag{19}$$

$$u(x) = 0, \qquad x \in \partial\Omega \tag{20}$$

is the Green's function integral operator

$$u(x) = \mathcal{G} f(x). \tag{21}$$

Lemma 2 *The operator $\mathcal{G}$ is a bounded and self–adjoint operator from $L^2(\Omega)$ into $H_0^2(\Omega)$. Moreover, it is a compact operator from $L^2(\Omega)$ into $H_0^1(\Omega)$, and more particularly, it is a compact operator from $H_0^1(\Omega)$ into $H_0^1(\Omega)$.*

Proof. A proof can be based on [16, §6.3, Thm. 5] together with the fact that the embedding of $H_0^2(\Omega)$ into $H_0^1(\Omega)$ is compact. The symmetry follows from the self–adjointness of the original problem (19)–(20). ■

We convert (16) to

$$(f,v) = \mathcal{A}(\mathcal{G}f, v), \qquad v \in H_0^1(\Omega), \quad f \in L^2(\Omega). \tag{22}$$

The problem (19)–(20) has the following variational reformulation: find $u \in H_0^1(\Omega)$ such that

$$\mathcal{A}(u,v) = \ell(v), \qquad \forall v \in H_0^1(\Omega). \tag{23}$$

This problem can be shown to have a unique solution u by using the Lax–Milgram Theorem to imply its existence; see [7, Thm. 8.3.4]. In addition,

$$\|u\|_1 \le \frac{1}{c_e} \|\ell\|$$

with $\|\ell\|$ denoting the operator norm for ℓ regarded as a linear functional on $H_0^1(\Omega)$.

2.2 The approximation scheme

Denote by Π_n the space of polynomials in d variables that are of degree $\leq n$: $p \in \Pi_n$ if it has the form

$$p(x) = \sum_{|i| \leq n} a_i x_1^{i_1} x_2^{i_2} \dots x_d^{i_d}$$

with i a multi–integer, $i = (i_1, \dots, i_d)$, and $|i| = i_1 + \cdots + i_d$. Over B_d, our approximation subspace is

$$\widetilde{\mathcal{X}}_n = \left\{ \left(1 - \|x\|_2^2\right) p(x) \mid p \in \Pi_n \right\} \tag{24}$$

with $\|x\|_2^2 = x_1^2 + \cdots + x_d^2$. The subspaces Π_n and $\widetilde{\mathcal{X}}_n$ have dimension

$$N \equiv N_n = \binom{n+d}{d}$$

However our problem (14) is defined over Ω, and thus we use a modification of $\widetilde{\mathcal{X}}_n$:

$$\mathcal{X}_n = \left\{ \psi(s) = \widetilde{\psi}(\Psi(s)) : \widetilde{\psi} \in \widetilde{\mathcal{X}}_n \right\} \tag{25}$$

The finite dimensional set $\mathcal{X}_n \subseteq H_0^1(\Omega)$. This set of functions is used in the initial definition of our numerical scheme and for its convergence analysis; but the simpler space $\widetilde{\mathcal{X}}_n$ is used in the actual implementation of the method.

To solve (23) (and thus (19)–(20)) approximately, we use the Galerkin method with trial space $\mathcal{X}_n$ to find $u_n \in \mathcal{X}_n$ for which

$$\mathcal{A}(u_n, v) = \ell(v), \qquad \forall v \in \mathcal{X}_n. \tag{26}$$

For the eigenvalue problem (1), find $u_n \in \mathcal{X}_n$ for which

$$\mathcal{A}(u_n, v) = \lambda(u_n, v), \qquad \forall v \in \mathcal{X}_n. \tag{27}$$

Write

$$u_n(s) = \sum_{j=1}^{N} \alpha_j \psi_j(s) \tag{28}$$

with $\{\psi_j\}_{j=1}^N$ a basis of $\mathcal{X}_n$. Then (27) becomes

$$\sum_{j=1}^{N} \alpha_j \mathcal{A}(\psi_j, \psi_i) = \lambda \sum_{j=1}^{N} \alpha_j (\psi_j, \psi_i), \quad i = 1, \dots, N \tag{29}$$

The coefficients can be related back to a polynomial basis for $\widetilde{\mathcal{X}}_n$ and to integrals over B_d. Let $\left\{\widetilde{\psi}_j\right\}$ denote the basis of $\widetilde{\mathcal{X}}_n$ corresponding to the basis

$\{\psi_j\}$ for $\mathcal{X}_n$. Using the transformation $s = \Phi(x)$,

$$\begin{aligned}(\psi_j, \psi_i) &= \int_\Omega \psi_j(s)\,\psi_i(s)\,ds \\ &= \int_{B_d} \widetilde{\psi}_j(x)\,\widetilde{\psi}_i(x)\,|\det J(x)|\,dx \end{aligned} \tag{30}$$

$$\begin{aligned}\mathcal{A}(\psi_j, \psi_i) &= \int_\Omega \left[\sum_{k,\ell=1}^{d} a_{k,\ell}(s)\,\frac{\partial\psi_j(s)}{\partial s_k}\frac{\partial\psi_i(s)}{\partial s_\ell} + \gamma(s)\psi_j(s)\psi_i(s)\right] ds \\ &= \int_\Omega \left[\{\nabla_s\psi_i(s)\}^{\mathrm{T}} A(s)\,\{\nabla_s\psi_j(s)\} + \gamma(s)\psi_j(s)\psi_i(s)\right] ds \\ &= \int_\Omega \left[\left\{K(\Phi(x))^{\mathrm{T}}\nabla_x\widetilde{\psi}_i(x)\right\}^{\mathrm{T}} A(\Phi(x))\left\{K(\Phi(x))^{\mathrm{T}}\nabla_x\widetilde{\psi}_j(x)\right\}\right. \\ &\qquad \left. +\widetilde{\gamma}(x)\widetilde{\psi}_j(x)\widetilde{\psi}_i(x)\right] |\det J(x)|\,dx \\ &= \int_{B_d} \left[\nabla_x\widetilde{\psi}_i(x)^{\mathrm{T}}\,\widetilde{A}(x)\nabla_x\widetilde{\psi}_j(x) + \widetilde{\gamma}(x)\widetilde{\psi}_i(x)\,\widetilde{\psi}_j(x)\right] |\det J(x)|\,dx \end{aligned}$$

with the matrix $\widetilde{A}(x)$ given in (12). With these evaluations of the coefficients, it is straightforward to show that (29) is equivalent to a Galerkin method for (12) using the standard inner product of $L^2(B_d)$ and the approximating subspace $\widetilde{\mathcal{X}}_n$.

2.3 Convergence analysis

The scheme (29) is implicitly a numerical approximation of the integral equation eigenvalue problem

$$\lambda\mathcal{G}u = u. \tag{31}$$

Lemma 3 *The numerical method (27) is equivalent to the Galerkin method approximation of the integral equation (31), with the Galerkin method based on the inner product $\mathcal{A}(\cdot,\cdot)$ for $H_0^1(\Omega)$.*

Proof. For the Galerkin solution of (31) we seek a function u_n in the form (28), and we force the residual to be orthogonal to $\mathcal{X}_n$. This leads to

$$\lambda\sum_{j=1}^{N}\alpha_j\mathcal{A}(\mathcal{G}\psi_j, \psi_i) = \sum_{j=1}^{N}\alpha_j\mathcal{A}(\psi_j, \psi_i) \tag{32}$$

for $i = 1, \ldots, N$. From (22), we have $\mathcal{A}(\mathcal{G}\psi_j, \psi_i) = (\psi_j, \psi_i)$, and thus

$$\lambda\sum_{j=1}^{N}\alpha_j(\psi_j, \psi_i) = \sum_{j=1}^{N}\alpha_j\mathcal{A}(\psi_j, \psi_i)$$

This is exactly the same as (29). ■

Let $\mathcal{P}_n$ be the orthogonal projection of $H_0^1(B)$ onto $\mathcal{X}_n$, based on the inner product $\mathcal{A}(\cdot,\cdot)$. Then (32) is the Galerkin approximation,

$$\mathcal{P}_n \mathcal{G} u_n = \frac{1}{\lambda} u_n, \quad u_n \in \mathcal{X}_n \tag{33}$$

for the integral equation eigenvalue problem (31). Much is known about such schemes, as we discuss below. The conversion of the eigenvalue problem (27) into the equivalent eigenvalue problem (33) is motivated by a similar idea used in Osborn [25].

The numerical solution of eigenvalue problems for compact integral operators has been studied by many people for over a century. With Galerkin methods, we note particularly the early work of Krasnoselskii [20, p. 178]. The book of Chatelin [14] presents and summarizes much of the literature on the numerical solution of such eigenvalue problems for compact operators. For our work we use the results given in [2], [3] for pointwise convergent operator approximations that are collectively compact.

We begin with some preliminary lemmas.

Lemma 4 *For suitable positive constants c_1 and c_2,*

$$c_1 \|\widetilde{v}\|_{H_0^1(B_d)} \leq \|v\|_{H_0^1(\Omega)} \leq c_2 \|\widetilde{v}\|_{H_0^1(B_d)}$$

for all functions $v \in H_0^1(\Omega)$, with $\widetilde{v}$ the corresponding function of (5). Thus, for a sequence $\{v_n\}$ in $H_0^1(\Omega)$,

$$v_n \to v \quad \text{in } H_0^1(\Omega) \iff \widetilde{v}_n \to \widetilde{v} \quad \text{in } H_0^1(B_d) \tag{34}$$

with $\{\widetilde{v}_n\}$ the corresponding sequence in $H_0^1(B_d)$.

Proof. Begin by noting that there is a 1-1 correspondence between $H_0^1(\Omega)$ and $H_0^1(B_d)$ based on using (4)–(6). Next,

$$\begin{aligned}
\|v\|_{H_0^1(\Omega)}^2 &= \int_\Omega \left[|\nabla v(s)|^2 + |v(s)|^2 \right] ds \\
&= \int_{B_d} \left[\left| \nabla \widetilde{v}(x)^{\mathrm{T}} J(x)^{-1} J(x)^{-\mathrm{T}} \nabla \widetilde{v}(x) \right| + |\widetilde{v}(x)|^2 \right] |\det J(x)| \, dx \\
&\leq \left[\max_{x \in B} |\det J(x)| \right] \max \left\{ \max_{x \in B} \|J(x)^{-1}\|^2, 1 \right\} \int_{B_d} \left[|\nabla \widetilde{v}(x)|^2 + |\widetilde{v}(x)|^2 \right] dx
\end{aligned}$$

$$\|v\|_{H_0^1(\Omega)} \leq c_2 \|\widetilde{v}\|_{H_0^1(B_d)} \tag{35}$$

for a suitable constant $c_2(\Omega)$. The reverse inequality, with the roles of $\|\widetilde{v}\|_{H_0^1(B_d)}$ and $\|v\|_{H_0^1(\Omega)}$ reversed, follows by an analogous argument. ■

Lemma 5 *The set $\cup_{n \geq 1} \mathcal{X}_n$ is dense in $H_0^1(\Omega)$.*

Proof. The set $\cup_{n\geq 1}\widetilde{\mathcal{X}}_n$ is dense in $H_0^1(B_d)$, a result shown in [5, see (15)]. We can then use the correspondence between $H_0^1(\Omega)$ and $H_0^1(B_d)$, given in Lemma 4, to show that $\cup_{n\geq 1}\mathcal{X}_n$ is dense in $H_0^1(\Omega)$. ■

Lemma 6 *The standard norm $\|\cdot\|_1$ on $H_0^1(\Omega)$ and the norm $\|v\|_{\mathcal{A}} = \sqrt{\mathcal{A}(v,v)}$ are equivalent in the topology they generate. More precisely,*

$$\sqrt{c_e}\|v\|_1 \leq \|v\|_{\mathcal{A}} \leq \sqrt{c_{\mathcal{A}}}\|v\|_1, \qquad v \in H_0^1(\Omega). \tag{36}$$

with the constants $c_{\mathcal{A}}$, c_e taken from (17) and (18), respectively. Convergence of sequences $\{v_n\}$ is equivalent in the two norms.

Proof. It is immediate from (18) and (17). ■

Lemma 7 *For the orthogonal projection operator $\mathcal{P}_n$,*

$$\mathcal{P}_n v \to v \qquad as \quad n \to \infty, \qquad for\ all\ v \in H_0^1(\Omega). \tag{37}$$

Proof. This follows from the definition of an orthogonal projection operator and using the result that $\cup_{n\geq 1}\mathcal{X}_n$ is dense in $H_0^1(\Omega)$. ■

Corollary 8 *For the integral operator $\mathcal{G}$,*

$$\|(I-\mathcal{P}_n)\mathcal{G}\| \to 0 \quad as \quad n \to \infty \tag{38}$$

using the norm for operators from $H_0^1(\Omega)$ into $H_0^1(\Omega)$.

Proof. Consider $\mathcal{G}$ and $\mathcal{P}_n$ as operators on $H_0^1(\Omega)$ into $H_0^1(\Omega)$. The result follows from the compactness of $\mathcal{G}$ and the pointwise convergence in (37); see [4, Lemma 3.1.2]. ■

Lemma 9 *$\{\mathcal{P}_n\mathcal{G}\}$ is collectively compact on $H_0^1(\Omega)$.*

Proof. This follows for all such families $\{\mathcal{P}_n\mathcal{G}\}$ with $\mathcal{G}$ compact on a Banach space $\mathcal{Y}$ and $\{\mathcal{P}_n\}$ pointwise convergent on $\mathcal{Y}$. To prove this requires showing

$$\{\mathcal{P}_n\mathcal{G}v \mid \|v\|_1 \leq 1,\, n \geq 1\}$$

has compact closure in $H_0^1(\Omega)$. This can be done by showing that the set is totally bounded. We omit the details of the proof. ■

Summarizing, $\{\mathcal{P}_n\mathcal{G}\}$ is a collectively compact family that is pointwise convergent on $H_0^1(\Omega)$. With this, the results in [2], [3] can be applied to (33) as a numerical approximation to the eigenvalue problem (31). We summarize the application of those results to (33).

Theorem 10 *Let λ be an eigenvalue for the problem (1)–(2), say of multiplicity ν, and let $\chi^{(1)}, \ldots, \chi^{(\nu)}$ be a basis for the associated eigenfunction subspace. Let $\varepsilon > 0$ be chosen such that there are no other eigenvalues of (1)–(2) within a distance ε of λ. Let σ_n denote the eigenvalue solutions of (27) that are within ε of λ. Then for all sufficiently large n, say $n \geq n_0$, the sum of the multiplicities of the approximating eigenvalues within σ_n equals ν. Moreover,*

$$\max_{\lambda_n \in \sigma_n} |\lambda - \lambda_n| \leq c \max_{1 \leq k \leq \nu} \| (I - \mathcal{P}_n) \chi^{(k)} \|_1 \tag{39}$$

Let u be an eigenfunction of (1)–(2) associated with λ. Let $\mathcal{W}_n$ be the direct sum of the eigenfunction subspaces associated with the eigenvalues $\lambda_n \in \sigma_n$, and let $\left\{u_n^{(1)}, \ldots, u_n^{(\nu)}\right\}$ be a basis for $\mathcal{W}_n$. Then there is a sequence

$$u_n = \sum_{k=1}^{\nu} \alpha_{n,k} u_n^{(k)} \in \mathcal{W}_n$$

for which

$$\|u - u_n\|_1 \leq c \max_{1 \leq k \leq \nu} \| (I - \mathcal{P}_n) \chi^{(k)} \|_1 \tag{40}$$

for some constant $c > 0$ dependent on λ.

Proof. This is a direct consequence of results in [2], [3], together with the compactness of $\mathcal{G}$ on $H_0^1(\Omega)$. It also uses the equivalence of norms given in (36). ■

The norms $\| (I - \mathcal{P}_n) \chi^{(k)} \|_1$ can be bounded using results from Ragozin [26], just as was done in [5]. We begin with the following result from [26]. The corresponding result that is needed with the Neumann problem can be obtained from [9].

Lemma 11 *Assume $w \in C^{k+2}\left(\overline{B}_d\right)$ for some $k > 0$, and assume $w|_{\partial B} = 0$. Then there is a polynomial $q_n \in \widetilde{\mathcal{X}}_n$ for which*

$$\|w - q_n\|_\infty \leq D(k,d)\, n^{-k} \left(n^{-1} \|w\|_{\infty,k+2} + \omega\left(w^{(k+2)}, 1/n\right)\right) \tag{41}$$

In this,

$$\|w\|_{\infty,k+2} = \sum_{|i| \leq k+2} \left\|\partial^i w\right\|_\infty$$

$$\omega(g, \delta) = \sup_{|x-y| \leq \delta} |g(x) - g(y)|$$

$$\omega\left(w^{(k+2)}, \delta\right) = \sum_{|i|=k+2} \omega\left(\partial^i w, \delta\right)$$

Theorem 12 *Recall the notation and assumptions of Theorem 10. Assume the eigenfunction basis functions* $\chi^{(k)} \in C^{m+2}(\Omega)$ *and assume* $\Phi \in C^{m+2}(B_d)$*, for some* $m \geq 1$*. Then*

$$\max_{\lambda_n \in \sigma_n} |\lambda - \lambda_n| = \mathcal{O}\left(n^{-m}\right)$$

$$\|u - u_n\|_1 = \mathcal{O}\left(n^{-m}\right)$$

Proof. Begin with (39)–(40). To obtain the bounds for $\| (I - \mathcal{P}_n) u^{(k)} \|_1$ given above using Lemma 11, refer to the argument given in [5]. ■

3 Implementation

Consider the implementation of the Galerkin method of (27) for the eigenvalue problem (1). We are to find the function $u_n \in \mathcal{X}_n$ satisfying (29). To do so, we begin by selecting a basis for Π_n that is orthonormal in $L^2(B_d)$, denoting it by $\{\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_N\}$, with $N \equiv N_n = \dim \Pi_n$. Choosing such an orthonormal basis is an attempt to have the matrix associated with the left side of the linear system in (29) be better conditioned. Next, let

$$\widetilde{\psi}_i(x) = \left(1 - \|x\|_2^2\right) \widetilde{\varphi}_i(x), \qquad i = 1, \ldots, N_n \tag{42}$$

to form a basis for $\widetilde{\mathcal{X}}_n$. As in (25), let $\{\psi_1, \ldots, \psi_N\}$ be the corresponding basis of $\mathcal{X}_n$.

We seek

$$u_n(s) = \sum_{j=1}^{N} \alpha_j \psi_j(s) \tag{43}$$

Then following the change of variable $s = \Phi(x)$, (29) becomes

$$\begin{aligned} \sum_{j=1}^{N} \alpha_j \int_{B_d} \left[\nabla \widetilde{\psi}_j(x)^{\mathrm{T}} \widetilde{A}(x) \nabla \widetilde{\psi}_i(x) + \widetilde{\gamma}(x) \widetilde{\psi}_j(x) \widetilde{\psi}_i(x)\right] |\det J(x)| \, dx \\ = \lambda \sum_{j=1}^{N} \alpha_j \int_{B_d} \widetilde{\psi}_j(x) \widetilde{\psi}_i(x) |\det J(x)| \, dx, \qquad i = 1, \ldots, N \end{aligned} \tag{44}$$

We need to calculate the orthonormal polynomials and their first partial derivatives; and we also need to approximate the integrals in the linear system. For an introduction to the topic of multivariate orthogonal polynomials, see Dunkl and Xu [15] and Xu [30]. For multivariate quadrature over the unit ball in $\mathbb{R}^d$, see Stroud [28].

3.1 The planar case

The dimension of Π_n is

$$N_n = \frac{1}{2}(n+1)(n+2) \tag{45}$$

For notation, we replace x with (x, y). How do we choose the orthonormal basis $\{\widetilde{\varphi}_\ell(x,y)\}_{\ell=1}^N$ for Π_n? Unlike the situation for the single variable case, there are many possible orthonormal bases over $B = D$, the unit disk in $\mathbb{R}^2$. We have chosen one that is particularly convenient for our computations. These are the "ridge polynomials" introduced by Logan and Shepp [22] for solving an image reconstruction problem. We summarize here the results needed for our work.

Let

$$\mathcal{V}_n = \{P \in \Pi_n : (P, Q) = 0 \quad \forall Q \in \Pi_{n-1}\}$$

the polynomials of degree n that are orthogonal to all elements of Π_{n-1}. Then the dimension of $\mathcal{V}_n$ is $n+1$; moreover,

$$\Pi_n = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n \tag{46}$$

It is standard to construct orthonormal bases of each $\mathcal{V}_n$ and to then combine them to form an orthonormal basis of Π_n using the latter decomposition. As an orthonormal basis of $\mathcal{V}_n$ we use

$$\widetilde{\varphi}_{n,k}(x,y) = \frac{1}{\sqrt{\pi}} U_n\left(x\cos(kh) + y\sin(kh)\right), \quad (x,y) \in D, \quad h = \frac{\pi}{n+1} \tag{47}$$

for $k = 0, 1, \ldots, n$. The function U_n is the Chebyshev polynomial of the second kind of degree n:

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad t = \cos\theta, \quad -1 \le t \le 1, \quad n = 0, 1, \ldots \tag{48}$$

The family $\{\widetilde{\varphi}_{n,k}\}_{k=0}^n$ is an orthonormal basis of $\mathcal{V}_n$. As a basis of Π_n, we order $\{\widetilde{\varphi}_{n,k}\}$ lexicographically based on the ordering in (47) and (46):

$$\{\widetilde{\varphi}_\ell\}_{\ell=1}^N = \{\widetilde{\varphi}_{0,0},\, \widetilde{\varphi}_{1,0},\, \widetilde{\varphi}_{1,1},\, \widetilde{\varphi}_{2,0},\, \ldots,\, \widetilde{\varphi}_{n,0},\, \ldots, \widetilde{\varphi}_{n,n}\}$$

Returning to (42), we define

$$\widetilde{\psi}_{n,k}(x,y) = \left(1 - x^2 - y^2\right)\widetilde{\varphi}_{n,k}(x,y) \tag{49}$$

To calculate the first order partial derivatives of $\widetilde{\psi}_{n,k}(x,y)$, we need $U_n'(t)$. The values of $U_n(t)$ and $U_n'(t)$ are evaluated using the standard triple recursion relations

$$\begin{aligned} U_{n+1}(t) &= 2tU_n(t) - U_{n-1}(t) \\ U_{n+1}'(t) &= 2U_n(t) + 2tU_n'(t) - U_{n-1}'(t) \end{aligned}$$

For the numerical approximation of the integrals in (44), which are over B being the unit disk, we use the formula

$$\int_B g(x,y)\,dx\,dy \approx \sum_{l=0}^{q}\sum_{m=0}^{2q} g\left(r_l, \frac{2\pi\, m}{2q+1}\right)\omega_l \frac{2\pi}{2q+1} r_l \tag{50}$$

Here the numbers ω_l are the weights of the $(q+1)$-point Gauss-Legendre quadrature formula on $[0,1]$. Note that

$$\int_0^1 p(x)dx = \sum_{l=0}^{q} p(r_l)\omega_l,$$

for all single-variable polynomials $p(x)$ with $\deg(p) \le 2q+1$. The formula (50) uses the trapezoidal rule with $2q+1$ subdivisions for the integration over $\overline{B}$ in the azimuthal variable. This quadrature (50) is exact for all polynomials $g \in \Pi_{2q}$. This formula is also the basis of the hyperinterpolation formula discussed in [18].

3.2 The three–dimensional case

In $\mathbb{R}^3$, the dimension of Π_n is

$$N_n = \binom{n+3}{3} = \frac{1}{6}(n+1)(n+2)(n+3)$$

Here we choose orthonormal polynomials on the unit ball as described in [15],

$$\begin{aligned}\widetilde{\varphi}_{m,j,\beta}(x) &= c_{m,j} p_j^{(0,m-2j+\frac{1}{2})}(2\|x\|^2-1) S_{\beta,m-2j}\left(\frac{x}{\|x\|}\right)\\ &= c_{m,j}\|x\|^{m-2j} p_j^{(0,m-2j+\frac{1}{2})}(2\|x\|^2-1) S_{\beta,m-2j}\left(\frac{x}{\|x\|}\right), \qquad (51)\\ j &= 0,\ldots,\lfloor m/2\rfloor, \quad \beta = 0,1,\ldots,2(m-2j), \quad m = 0,1,\ldots,n\end{aligned}$$

Here $c_{m,j} = 2^{\frac{5}{4}+\frac{m}{2}-j}$ is a constant, and $p_j^{(0,m-2j+\frac{1}{2})}$, $j \in \mathbb{N}_0$, are the normalized Jabobi polynomials which are orthonormal on $[-1,1]$ with respect to the inner product

$$(v,w) = \int_{-1}^{1} (1+t)^{m-2j+\frac{1}{2}} v(t)w(t)\,dt,$$

see for example [1], [17]. The functions $S_{\beta,m-2j}$ are spherical harmonic functions, and they are given in spherical coordinates by

$$S_{\beta,k}(\phi,\theta) = \widetilde{c}_{\beta,k}\begin{cases} \cos(\frac{\beta}{2}\phi) T_k^{\frac{\beta}{2}}(\cos\theta), & \beta \text{ even}\\ \sin(\frac{\beta+1}{2}\phi) T_k^{\frac{\beta+1}{2}}(\cos\theta), & \beta \text{ odd}\end{cases}$$

The constant $\widetilde{c}_{\beta,k}$ is chosen in such a way that the functions are orthonormal on the unit sphere S^2 in $\mathbb{R}^3$:

$$\int_{S^2} S_{\beta,k}(x)\, S_{\widetilde{\beta},\widetilde{k}}(x)\, dS = \delta_{\beta,\widetilde{\beta}}\, \delta_{k,\widetilde{k}}$$

The functions T_k^l are the associated Legendre polynomials, see [19], [23]. According to (42) we define the basis for our space of trial functions by

$$\widetilde{\psi}_{m,j,\beta}(x) = (1 - \|x\|^2)\widetilde{\varphi}_{m,j,\beta}(x)$$

and we can order the basis lexicographically. To calculate all of the above functions we can use recursive algorithms similar to the one used for the Chebyshev polynomials. These algorithms also allow the calculation of the derivatives of each of these functions, see [17], [32]

For the numerical approximation of the integrals in (44) we use a quadrature formula for the unit ball B

$$\begin{aligned}\int_B g(x)\, dx &= \int_0^1 \int_0^{2\pi} \int_0^{\pi} \widetilde{g}(r,\theta,\phi)\, r^2 \sin(\phi)\, d\phi\, d\theta\, dr \approx Q_q[g] \\ Q_q[g] &:= \sum_{i=1}^{2q} \sum_{j=1}^{q} \sum_{k=1}^{q} \frac{\pi}{q}\, \omega_j\, \nu_k \widetilde{g}\left(\frac{\zeta_k + 1}{2}, \frac{\pi\, i}{2q}, \arccos(\xi_j)\right) \end{aligned} \tag{52}$$

Here $\widetilde{g}(r,\theta,\phi) = g(x)$ is the representation of g in spherical coordinates. For the θ integration we use the trapezoidal rule, because the function is $2\pi-$periodic in θ. For the r direction we use the transformation

$$\begin{aligned}\int_0^1 r^2 v(r)\, dr &= \int_{-1}^1 \left(\frac{t+1}{2}\right)^2 v\left(\frac{t+1}{2}\right) \frac{dt}{2} \\ &= \frac{1}{8}\int_{-1}^1 (t+1)^2 v\left(\frac{t+1}{2}\right)\, dt \\ &\approx \sum_{k=1}^{q} \underbrace{\frac{1}{8}\nu_k'}_{=:\nu_k} v\left(\frac{\zeta_k + 1}{2}\right)\end{aligned}$$

where the ν_k' and ζ_k are the weights and the nodes of the Gauss quadrature with q nodes on $[-1,1]$ with respect to the inner product

$$(v,w) = \int_{-1}^1 (1+t)^2 v(t) w(t)\, dt$$

The weights and nodes also depend on q but we omit this index. For the ϕ direction we use the transformation

$$\begin{aligned}\int_0^{\pi} \sin(\phi) v(\phi)\, d\phi &= \int_{-1}^1 v(\arccos(\phi))\, d\phi \\ &\approx \sum_{j=1}^{q} \omega_j v(\arccos(\xi_j))\end{aligned}$$

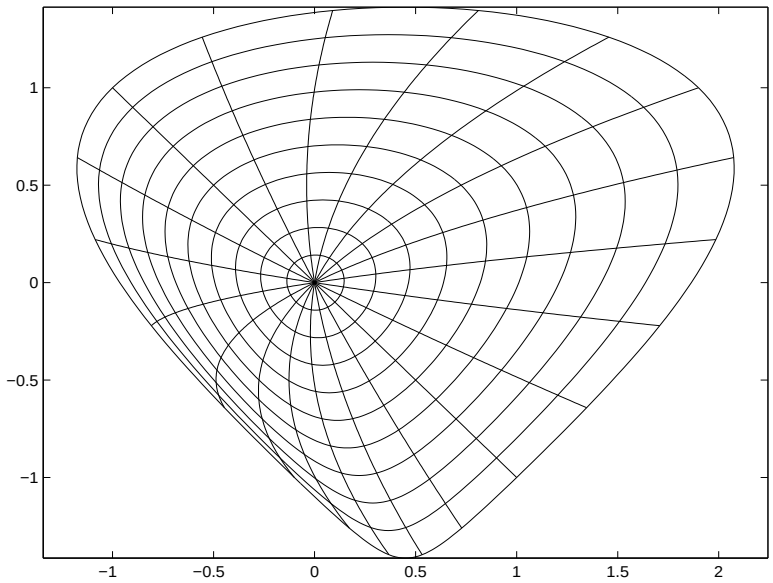

Figure 1: Images of (55), with $a = 0.5$, for lines of constant radius and constant azimuth on the unit disk.

where the ω_j and ξ_j are the nodes and weights for the Gauss–Legendre quadrature on $[-1, 1]$. For more information on this quadrature rule on the unit ball in $\mathbb{R}^3$, see [28].

Finally we need the gradient in Cartesian coordinates to approximate the integral in (44), but the function $\widetilde{\varphi}_{m,j,\beta}(x)$ in (51) is given in spherical coordinates. Here we simply use the chain rule, with $x = (x, y, z)$,

$$\begin{aligned}\frac{\partial}{\partial x}v(r,\theta,\phi) = \frac{\partial}{\partial r}v(r,\theta,\phi)\cos(\theta)\sin(\phi) - \frac{\partial}{\partial \theta}v(r,\theta,\phi)\frac{\sin(\theta)}{r\sin(\phi)} \\ + \frac{\partial}{\partial \phi}v(r,\theta,\phi)\frac{\cos(\theta)\cos(\phi)}{r}\end{aligned}$$

and similarly for $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$.

4 Numerical example

Our programs are written in MATLAB. The transformations have been so chosen that we can invert explicitly the mapping Φ, to be able to better construct our test examples. This is not needed when applying the method; but it simplified the construction of our test cases. The eigenvalue problem being solved is

$$Lu(\mathbf{s}) \equiv -\Delta u = \lambda u(\mathbf{s}), \qquad \mathbf{s} \in \Omega \subseteq \mathbb{R}^d \tag{53}$$

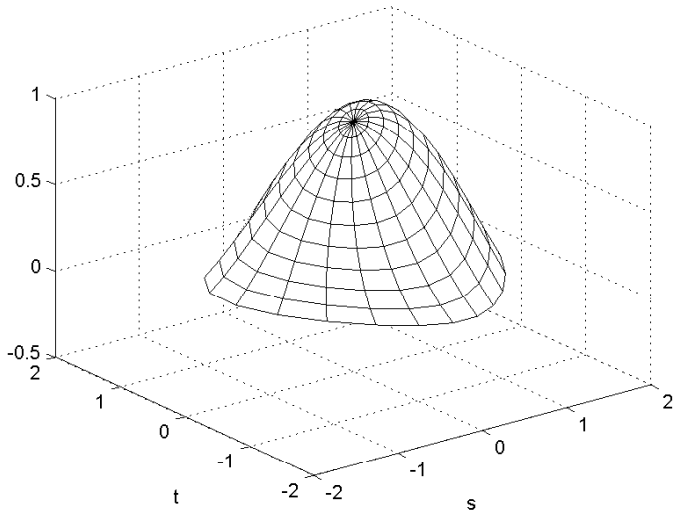

Figure 2: Eigenfunction corresponding to the approximate eigenvalue $\lambda_1 \doteq 2.96185$.

which corresponds to choosing $A = I$. Then we need to calculate

$$\widetilde{A}(\mathbf{x}) = J(\mathbf{x})^{-1} J(\mathbf{x})^{-\mathrm{T}} \tag{54}$$

4.1 The planar case

For our variables, we replace a point $x \in B_2$ with (x, y), and we replace a point $s \in \Omega$ with (s, t). Define the mapping $\Phi : \overline{B}_2 \to \overline{\Omega}$ by $(s, t) = \Phi(x, y)$,

$$\begin{aligned} s &= x - y + ax^2 \\ t &= x + y \end{aligned} \tag{55}$$

with $0 < a < 1$. It can be shown that Φ is a 1-1 mapping from the unit disk $\overline{B}$. In particular, the inverse mapping $\Psi : \overline{\Omega} \to \overline{B}$ is given by

$$\begin{aligned} x &= \frac{1}{a}\left[-1 + \sqrt{1 + a(s+t)}\right] \\ y &= \frac{1}{a}\left[at - \left(-1 + \sqrt{1 + a(s+t)}\right)\right] \end{aligned} \tag{56}$$

In Figure 1, we give the images in $\overline{\Omega}$ of the circles $r = j/10$, $j = 1, \ldots, 10$ and the azimuthal lines $\theta = j\pi/10$, $j = 1, \ldots, 20$.

The following information is needed when implementing the transformation from $-\Delta u = \lambda u$ on Ω to a new equation on B_2:

$$D\Phi = J(x, y) = \begin{pmatrix} 1 + 2ax & -1 \\ 1 & 1 \end{pmatrix}$$

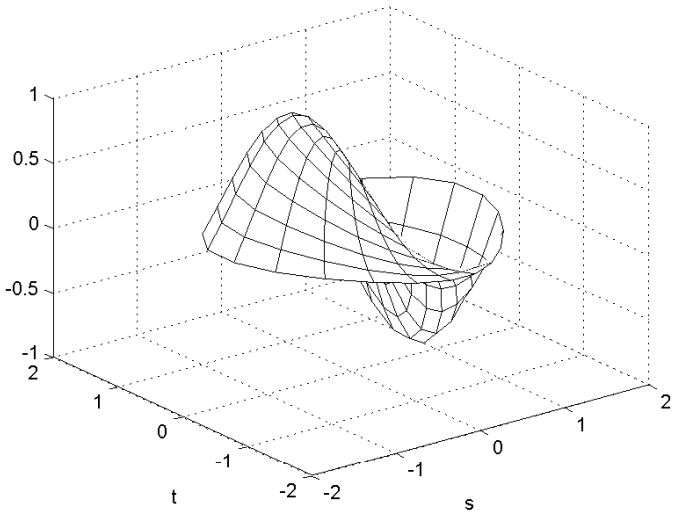

Figure 3: Eigenfunction corresponding to the approximate eigenvalue $\lambda_2 \doteq 7.24761$.

$$\det(J) = 2(1+ax)$$

$$J(x)^{-1} = \frac{1}{2(1+ax)} \begin{pmatrix} 1 & 1 \\ -1 & 1+2ax \end{pmatrix}$$

$$A = J(x)^{-1} J(x)^{-\mathrm{T}} = \frac{1}{2(1+ax)^2} \begin{pmatrix} 1 & ax \\ ax & 2a^2x^2+2ax+1 \end{pmatrix}$$

We give an example for this region Ω with $a = 0.5$. Figures 2 and 3 contain the computed eigenfunctions for the two smallest eigenvalues; these are based on the degree $n = 8$ approximation.

Because the true eigenfunctions and eigenvalues are unknown for almost all cases (with the unit ball as an exception), we used other methods for studying experimentally the rate of convergence. Let $\lambda_n^{(k)}$ denote the value of the k^{th} eigenvalue based on the degree n polynomial approximation, with the eigenvalues taken in increasing order. Let $u_n^{(k)}$ denote a corresponding eigenfunction,

$$\widetilde{u}_n^{(k)}(x) = \sum_{j=1}^{N_n} \alpha_j^{(n)} \widetilde{\psi}_j(x)$$

with $\alpha^{(n)} \equiv \left[\alpha_1^{(n)}, \ldots, \alpha_N^{(n)}\right]$ the eigenvector of (44) associated with the eigenvalue $\lambda_n^{(k)}$. We normalize the eigenvectors by requiring $\|\alpha^{(n)}\|_\infty = 1$. Define

$$\Lambda_n = \left|\lambda_{n+1}^{(k)} - \lambda_n^{(k)}\right|$$

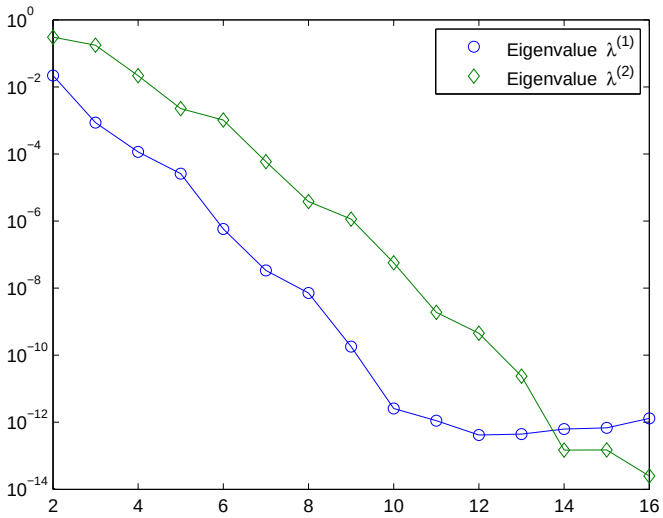

Figure 4: The values of $\left|\lambda_{n+1}^{(k)} - \lambda_n^{(k)}\right|$ for $k = 1, 2$ for increasing degree n.

$$D_n = \|u_{n+1}^{(k)} - u_n^{(k)}\|_\infty$$

Figures 4 and 5 show the decrease, respectively, of Λ_n and D_n as n increases. In both cases, we use a semi-log scale. Also, consider the residual

$$R_n^{(k)} = -\Delta u_n^{(k)} - \lambda_n^{(k)} u_n^{(k)}$$

Figure 6 shows the decrease of $\|R_n^{(k)}\|_\infty$, again on a semi-log scale.

These numerical results all indicate an exponential rate of convergence as a function of the degree n of the approximations $\left\{\lambda_n^{(k)} : n \geq 1\right\}$ and $\left\{u_n^{(k)} : n \geq 1\right\}$. In Figure 4, the maximum accuracy for $\lambda^{(1)}$ appears to have been found with the degree $n = 12$, approximately. For larger degrees, rounding errors dominate. We also see that the accuracy for the first eigenvalue-eigenfunction pair is better than that for the second such pair.

4.2 The three–dimensional case

Here we consider the problem of finding eigenvalues and eigenfunctions for the Neumann problem in $\Omega \subset \mathbb{R}^3$:

$$\begin{aligned} -\Delta u(s) &= \lambda u(s), \quad s \in \Omega \\ \frac{\partial u(s)}{\partial n} &= 0, \qquad s \in \partial\Omega \end{aligned} \tag{57}$$

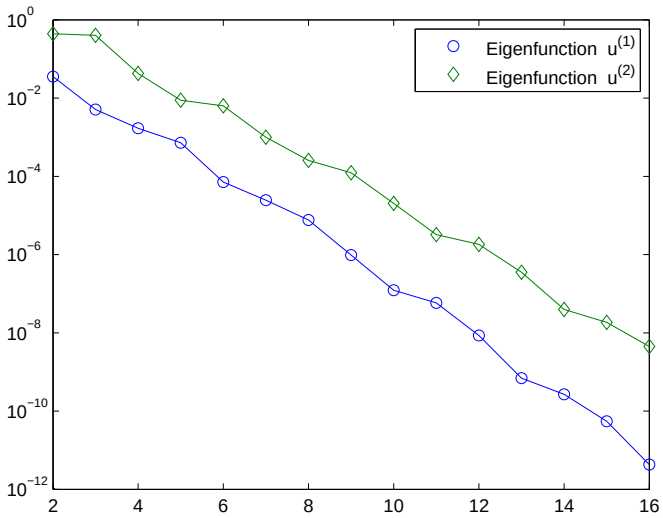

Figure 5: The values of $\|u_{n+1}^{(k)} - u_n^{(k)}\|_\infty$ for $k = 1, 2$ for increasing degree n.

Problem (57) is equivalent to

$$\begin{aligned} -\Delta u(s) + u(s) &= (\lambda + 1)u(s), \quad s \in \Omega \\ \frac{\partial u(s)}{\partial n} &= 0, \qquad\qquad\quad s \in \partial\Omega \end{aligned} \tag{58}$$

and $-\Delta + I : D_N \mapsto L^2(\Omega)$ is an invertible self–adjoint operator with

$$D_L = \left\{ u \in H^2(\Omega) \;\middle|\; \frac{\partial u(s)}{\partial n} = 0,\ s \in \partial\Omega \right\}$$

So there is a continuous solution operator $G : L^2(\Omega) \mapsto D_L$, such that

$$(-\Delta + I) \circ G|_{L^2(\Omega)} = I$$

with I the identity operator on $L^2(\Omega)$. If we consider $G : H^1(\Omega) \mapsto H^1(\Omega)$, then G is a compact operator, because of the compact imbedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ or $H^2(\Omega) \hookrightarrow H^1(\Omega)$; see [21] or [29].

We follow now Section 2.1 to present the variational framework. A solution of the inhomogeneous problem

$$Lu = f, \qquad f \in L^2(\Omega) \tag{59}$$

satisfies

$$\int_\Omega \left(-\sum_{j=1}^{3} \frac{\partial^2}{\partial s_j^2} u(s) + u(s) \right) v(s)\, ds = \int_\Omega f(s) v(s)\, ds \quad \text{for all } v \in H^1(\Omega)$$

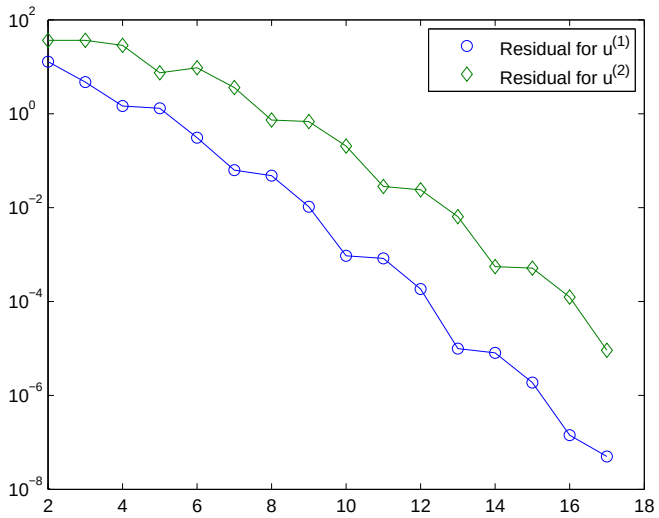

Figure 6: The values of $\|R_n^{(k)}\|_\infty$ for $k = 1, 2$ for increasing degree n.

Applying integration by parts and using the fact that the normal derivative of $u \in D_L$ is zero on $\partial\Omega$ we derive

$$\int_\Omega \nabla_s u(s) \nabla_s v(s) + u(s)v(s)\, ds = \int_\Omega f(s)v(s)\, ds \quad \text{for all } v \in H^1(\Omega)$$

We denote the left hand side of this equation by $\mathcal{A}(u, v)$ and from the Cauchy–Schwartz inequality we derive

$$\mathcal{A}(u, v) \le \|u\|_{H^1(\Omega)} \, \|v\|_{H^1(\Omega)}$$

and we have the equality

$$\mathcal{A}(u, u) = \|u\|^2_{H^1(\Omega)}$$

Because we assumed that the boundary $\partial\Omega$ is at least C^2, regularity theory shows that a solution $u \in H^1(\Omega)$ of the variational problem

$$\mathcal{A}(u, v) = (f, v) \quad \text{for all } v \in H^1(\Omega) \tag{60}$$

fulfills $u \in D_L$; see again [21] or [29]. So the problems (59) and (60) are equivalent.

Instead of (58) we consider the equivalent variational problem to find $u \in H^1(\Omega)$ which solves

$$\mathcal{A}(u, v) = (\lambda + 1) \int_\Omega u(s)v(s)\, ds \quad \text{for all } v \in H^1(\Omega)$$

and this is equivalent to

$$\int_\Omega \nabla_s u(s)\nabla_s v(s)\,ds = \lambda \int_\Omega u(s)v(s)\,ds \quad \text{for all } v \in H^1(\Omega) \tag{61}$$

Equation (61) is the starting point for our numerical approximation scheme, see also 27. First we transfer equation (61) to an equation on the domain B_3 with the help of a transformation $\Phi : B_3 \mapsto \Omega$. So (61) becomes

$$\begin{aligned}\int_{B_3} &\nabla_x \widetilde{u}(x)\widetilde{A}(x)\nabla_x \widetilde{v}(x)|\det(J(x)|\,dx \\ &= \lambda \int_{B_3} \widetilde{u}(x)\widetilde{v}(x)|\det(J(x)|\,dx \quad \text{for all } \widetilde{v} \in H^1(B_3)\end{aligned} \tag{62}$$

where $\widetilde{A}(x) = J(x)^{-1}J(x)^{-T}$; see (5)–(7) for the definition of the functions and $J(x)$. According to Section 2.2 we need a sequence of subspaces $\widetilde{\mathcal{X}}_n \subset \widetilde{\mathcal{X}}_{n+1} \subset H^1(B_3)$ with

$$\overline{\bigcup_{n=1}^{\infty} \widetilde{\mathcal{X}}_n} = H_1(B_3)$$

Because there are no boundary conditions imposed on $H^1(B_3)$ we can use

$$\mathcal{X}_n = \{p(x) \mid p \in \Pi_n\}$$

where Π_n is the space of polynomials in 3 variables of degree n or less. As a basis we choose

$$\{\widetilde{\varphi}_i(x) \mid i = 1, \ldots, N_n\}, \quad N_n = \binom{n+3}{3}$$

where $\{\widetilde{\varphi}_i\}$ is an enumeration of the orthogonal basis $\{\widetilde{\varphi}_{m,j,\beta}\}$ given in (51). To approximate the solutions $\widetilde{u}(x)$ of (62) we use

$$\widetilde{u}_n^{(i)}(x) = \sum_{j=1}^{N_n} \alpha_j^{(i)} \widetilde{\varphi}_j(x), : i =, \ldots, N_n$$

and the coefficients $\alpha_j^{(i)}$ for the eigenvalue approximation $\lambda_n^{(i)}$ are given as solutions of the finite eigenvalue problem

$$\begin{aligned}\sum_{j=1}^{N_n} &\left(\int_{B_3} \nabla_x \widetilde{\varphi}_j(x)\widetilde{A}(x)\nabla_x \widetilde{\varphi}_k(x)|\det(J(x))|\,dx \right) \alpha_j^{(i)} \\ &= \lambda_n^{(i)} \sum_{j=1}^{N_n} \left(\int_{B_3} \widetilde{\varphi}_j(x)\widetilde{\varphi}_k(x)|\det(J(x))|\,dx \right) \alpha_j^{(i)}, \quad k = 1, \ldots, N_n\end{aligned} \tag{63}$$

The functions $\nabla_x \widetilde{\varphi}_j(x)$ can be calculated explicitly and all integrals in formula (63) are approximated by the quadrature formula (52) with $q = n$. The convergence analysis of Section 2.3 can be used without any modifications.

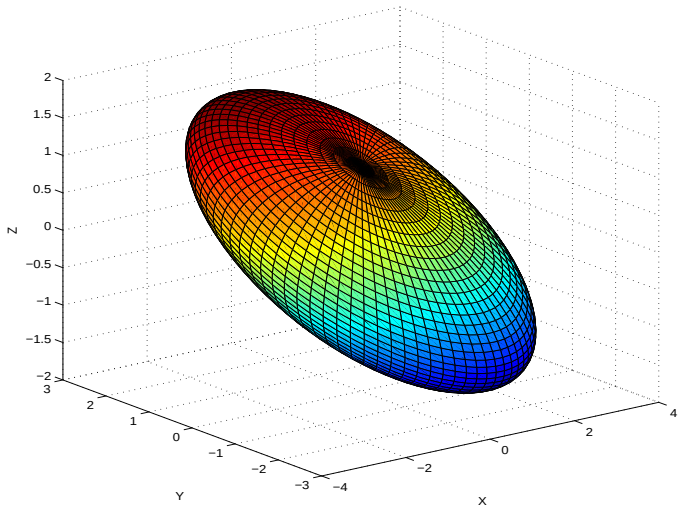

Figure 7: The boundary of Ω_1

To test our method we use two different domains. Let B_3 denote the closed unit ball in $\mathbb{R}^3$. The domain $\Omega_1 = \Phi_1(B_3)$ is given by

$$s = \Phi_1(x) \equiv \begin{pmatrix} x_1 - 3x_2 \\ 2x_1 + x_2 \\ x_1 + x_2 + x_3 \end{pmatrix}$$

so B_3 is transformed to an ellipsoid Ω_1; see Figure 7. The domain Ω_2 is given by

$$\Phi_2 \begin{pmatrix} \rho \\ \phi \\ \theta \end{pmatrix} = \begin{pmatrix} (1 - t(\rho))\rho + t(\rho)S(\phi, \theta) \\ \phi \\ \theta \end{pmatrix} \tag{64}$$

where we used polar coordinates $(\rho, \phi, \theta) \in [0, 1] \times [0, 2\pi] \times [0, \pi]$ to define the mapping Φ_2. Here the function $S : S^2 = \partial B_3 \mapsto (1, \infty)$ is a function which determines the boundary of a star shaped domain Ω_2. The restriction $S(\phi, \theta) > 1$ guarantees that Φ_2 is injective, and this can always be assumed after a suitable scaling of Ω_2. For our numerical example we use

$$S(\theta, \phi) = 2 + \frac{3}{4}\cos(2\phi)\sin(\theta)^2(7\cos(\theta)^2 - 1)$$

Finally the function t is defined by

$$t(\rho) \equiv \begin{cases} 0, & 0 \le \rho \le \frac{1}{2}, \\ 2^5(\rho - \frac{1}{2})^5, & \frac{1}{2} < \rho \le 1. \end{cases}$$

Table 1: Numerical results for Ω_1, $h = 0.0001$ to approximate $R_n^{(i)}$

n	N_n	$\lvert\lambda_n^{(1)} - \lambda_{15}^{(1)}\rvert$	$\lvert\lambda_n^{(2)} - \lambda_{15}^{(2)}\rvert$	$\angle(u_n^{(1)}, u_{15}^{(1)})$	$\angle(u_n^{(2)}, u_{15}^{(2)})$	$R_n^{(1)}$	$R_n^{(2)}$
1	4	$4.26E-2$	$1.00E-1$	$9.93E-2$	$1.31E-1$	$1.45E-1$	$2.63E-1$
2	10	$4.26E-2$	$1.00E-1$	$9.93E-2$	$1.31E-1$	$1.45E-1$	$2.63E-1$
3	20	$1.42E-4$	$5.67E-4$	$5.47E-3$	$1.01E-2$	$2.28E-2$	$5.15E-2$
4	35	$1.42E-4$	$5.67E-4$	$5.47E-3$	$1.01E-2$	$2.28E-2$	$5.15E-2$
5	56	$1.06E-7$	$8.38E-7$	$1.04E-4$	$2.72E-4$	$1.22E-3$	$3.54E-3$
6	84	$1.06E-7$	$8.38E-7$	$1.04E-4$	$2.72E-4$	$1.22E-3$	$3.54E-3$
7	120	$2.53E-11$	$4.31E-10$	$1.24E-6$	$4.85E-6$	$3.02E-5$	$1.08E-4$
8	165	$2.53E-11$	$4.31E-10$	$1.24E-6$	$4.85E-6$	$3.02E-5$	$1.08E-4$
9	220	$2.22E-11$	$6.78E-14$	0	$6.32E-8$	$4.25E-7$	$1.80E-6$
10	286	$4.47E-11$	$1.81E-13$	0	$5.77E-8$	$4.21E-7$	$1.80E-6$
11	364	$1.84E-13$	$5.19E-13$	0	0	$1.48E-8$	$4.15E-8$
12	455	$2.07E-13$	$1.18E-13$	0	0	$2.21E-9$	$1.88E-8$
13	560	$1.52E-13$	$1.91E-13$	0	0	$5.81E-9$	$3.43E-8$
14	680	$4.64E-13$	$5.56E-14$	0	0	$1.21E-8$	$4.26E-8$

Table 2: Numerical results for Ω_2

n	N_n	$\lvert\lambda_n^{(1)} - \lambda_{15}^{(1)}\rvert$	$\lvert\lambda_n^{(2)} - \lambda_{15}^{(2)}\rvert$	$\angle(u_n^{(1)}, u_{15}^{(1)})$	$\angle(u_n^{(2)}, u_{15}^{(2)})$
1	4	$3.60E-1$	$3.21E-1$	$2.86E-1$	$3.31E-1$
2	10	$3.60E-1$	$3.21E-1$	$2.86E-1$	$3.31E-1$
3	20	$8.16E-2$	$8.53E-2$	$8.32E-2$	$1.05E-1$
4	35	$8.16E-2$	$8.53E-2$	$8.32E-2$	$1.05E-1$
5	56	$1.99E-2$	$2.27E-2$	$2.84E-2$	$3.11E-2$
6	84	$1.99E-2$	$2.27E-2$	$2.84E-2$	$3.11E-2$
7	120	$1.48E-2$	$1.56E-2$	$2.49E-2$	$2.71E-2$
8	165	$1.48E-2$	$1.56E-2$	$2.49E-2$	$2.71E-2$
9	220	$4.77E-3$	$5.96E-3$	$1.14E-2$	$1.47E-2$
10	286	$4.77E-3$	$5.96E-3$	$1.14E-2$	$1.47E-2$
11	364	$8.34E-4$	$1.28E-3$	$3.25E-3$	$4.76E-3$
12	455	$8.34E-4$	$1.28E-3$	$3.25E-3$	$4.76E-3$
13	560	$1.88E-4$	$2.55E-4$	$1.33E-3$	$1.62E-3$
14	680	$1.88E-4$	$2.55E-4$	$1.33E-3$	$1.62E-3$

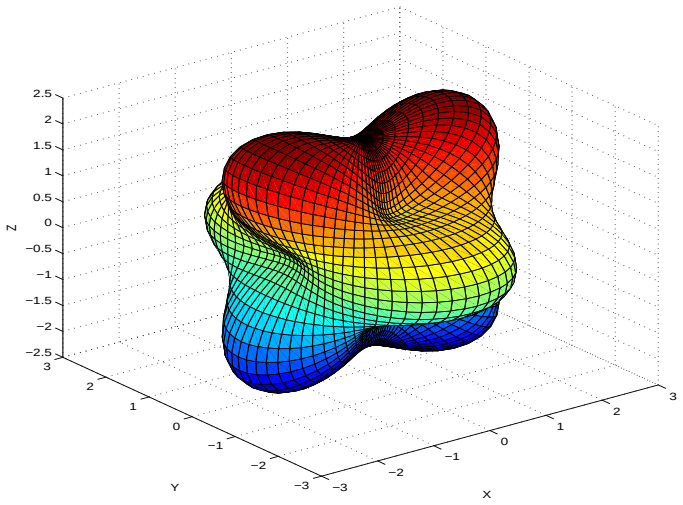

Figure 8: A view of $\partial\Omega_2$

where the exponent 5 implies $\Phi_2 \in C^4(B_1(0))$. See [6] for a more detailed description of Φ_2; one perspective of the surface Ω_2 is shown in Figure 8.

For each domain we calculate the approximate eigenvalues $\lambda_n^{(i)}$, $\lambda_n^{(0)} = 0 < \lambda_n^{(1)} \leq \lambda_n^{(2)} \leq \ldots$ and eigenfunctions $u_n^{(i)}$, $i = 1, \ldots, N_n$, for the degrees $n = 1, \ldots, 15$ (here we do not indicate dependence on the domain Ω). To analyze the convergence we calculate several numbers. First we estimate the speed of convergence for the first two eigenvalues by calculating $|\lambda_{15}^{(i)} - \lambda_n^{(i)}|$, $i = 1, 2$, $n = 1, \ldots, 14$. Then to estimate the speed of convergence of the eigenfunctions we calculate the angle (in $L^2(\Omega)$) between the current approximation and the most accurate approximation $\angle(u_n^{(i)}, u_{15}^{(i)})$, $i = 1, 2$, $n = 1, \ldots, 14$. Finally, an independent estimate of the quality of our approximation is given by

$$R_n^{(i)} \equiv | - \Delta u_n^{(i)}(s) - \lambda_n^{(i)} u(s)|$$

where we use only one $s \in \Omega$, given by $\Phi(1/10, 1/10, 1/10)$. To approximate the Laplace operator we use a second order difference scheme with $h = 0.0001$ for Ω_1 and $h = 0.01$ for Ω_2. The reason for the latter choice of h is that our approximations for the eigenfunctions on Ω_2 are only accurate up three to four digits, so if we divide by h^2 the discretization errors are magnified to the order of 1.

The numerical results for Ω_1 are given in table 4.2. The graphs in Figures 9–11, seem to indicate exponential convergence. For the graphs of $\angle(u_n^{(i)}, u_{15}^{(i)})$, see Figure 10. We remark that we use the function $\arccos(x)$ to calculate the angle, and for $n \approx 9$ the numerical calculations give $x = 1$, so the calculated

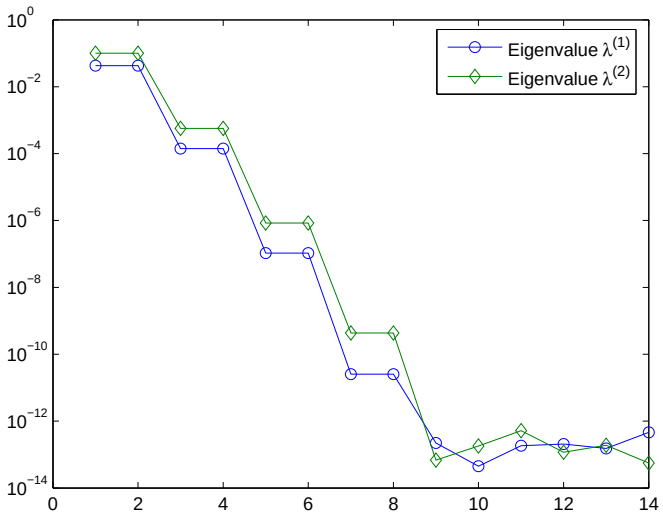

Figure 9: Ω_1: errors $\left|\lambda_{15}^{(i)} - \lambda_n^{(i)}\right|$ for the calculation of the first two eigenvalues $\lambda^{(i)}$

angle becomes 0. For the approximation of $R_n^{(i)}$ one has to remember that we use a difference method of order $O(h^2)$ to approximate the Laplace operator, so we can not expect any result better than 10^{-8} if we use $h = 0.0001$.

As we expect, the approximations for Ω_2 with the transformation Φ_2 present a bigger problem for our method. Still from the graphs in Figure 12 and 13 we might infer that the convergence is exponential, but with a smaller exponent than for Ω_1. Because $\Phi_2 \in C^4(B_3)$ we know that the transformed eigenfunctions on B_3 are in general only C^4, so we can only expect a convergence of $O(n^{-4})$. The values of n which we use are too small to show what we believe is the true behavior of the $R_n^{(i)}$, although the values for $n = 10 \ldots 14$ seem to indicate some convergence of the type we would expect.

The poorer convergence for Ω_2 as compared to Ω_1 illustrates a general problem. When defining a surface $\partial\Omega$ by giving it as the image of a 1-1 mapping from the unit sphere S^2 into $\mathbb{R}^3$, how does one extend it to a smooth mapping from the unit ball to Ω? The mapping in (64) is smooth, but it has large changes in its derivatives, and this affects the rate of convergence of our spectral method. We are working at present on this problem, developing a numerical method to find a well-behaved polynomial mapping Φ when given only its restriction to S^2.

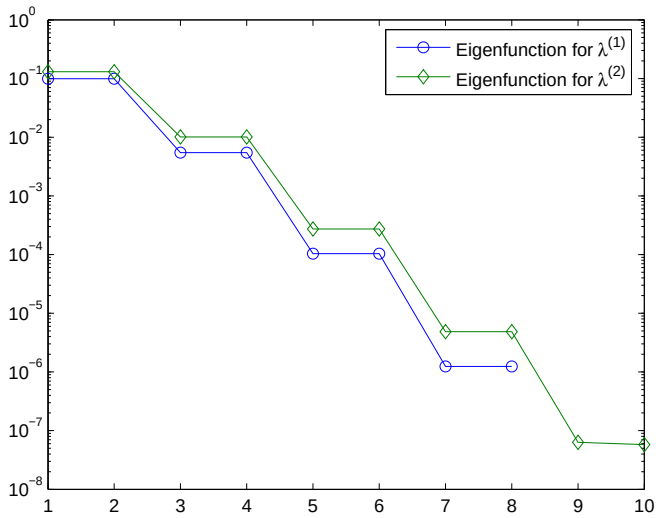

Figure 10: Ω_1: angles $\angle\left(u_n^{(i)}, u_{15}^{(i)}\right)$ between the approximate eigenfunction $u_n^{(i)}$ and our most accurate approximation $u_{15}^{(i)} \approx u^{(i)}$.

References

[1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, Inc., New York, 1965.

[2] K. Atkinson. The numerical solution of the eigenvalue problem for compact integral operators, *Trans. Amer. Math. Soc.* **129** (1967), 458-465.

[3] K. Atkinson. Convergence rates for approximate eigenvalues of compact integral operators, *SIAM J. Num. Anal.* **12** (1975), 213-222.

[4] K. Atkinson. *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge Univ. Press, 1997.

[5] K. Atkinson, D. Chien, and O. Hansen. A Spectral Method for Elliptic Equations: The Dirichlet Problem, *Advances in Computational Mathematics,* DOI: 10.1007/s10444-009-9125-8, to appear.

[6] K. Atkinson, D. Chien, and O. Hansen. A Spectral Method for Elliptic Equations: The Neumann Problem, submitted for publication. Available from http://arxiv.org/PS_cache/arxiv/pdf/0907/0907.1270v1.pdf.

[7] K. Atkinson and W. Han. *Theoretical Numerical Analysis: A Functional Analysis Framework*, 2nd ed., Springer-Verlag, New York, 2005.

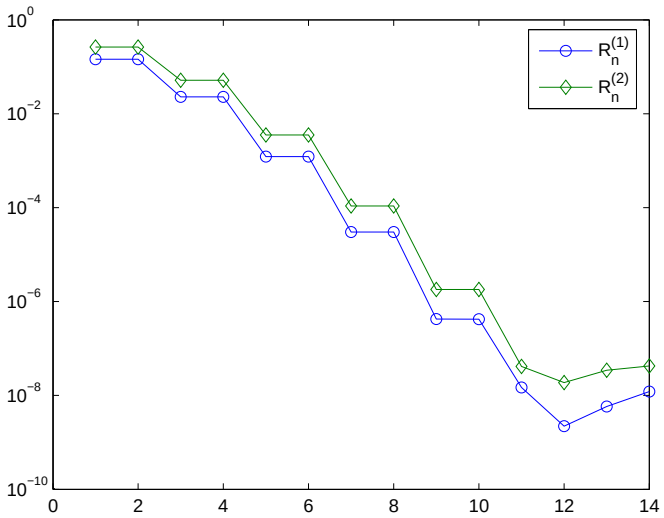

Figure 11: Ω_1: errors $\left|-\Delta u_n^{(i)}(s) - \lambda_n^{(i)} u_n^{(i)}(s)\right|$

[8] K. Atkinson and O. Hansen. Solving the nonlinear Poisson equation on the unit disk, *Journal of Integral Equations & Applications,* **17** (2005), 223-241.

[9] T. Bagby, L. Bos, and N. Levenberg, Multivariate simultaneous approximation, *Constructive Approximation*, **18** (2002), pp. 569-577.

[10] J. Boyd. *Chebyshev and Fourier Spectral Methods*, 2nd ed., Dover Pub., New York, 2000.

[11] S. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*, 2nd ed., Springer-Verlag, New York, 2002.

[12] C. Canuto, A. Quarteroni, My. Hussaini, and T. Zang, *Spectral Methods in Fluid Mechanics*, Springer-Verlag, 1988.

[13] C. Canuto, A. Quarteroni, My. Hussaini, and T. Zang, *Spectral Methods - Fundamentals in Single Domains*, Springer-Verlag, 2006.

[14] F. Chatelin. *Spectral Approximation of Linear Operators*, Academic Press, 1983.

[15] C. Dunkl and Y. Xu. *Orthogonal Polynomials of Several Variables*, Cambridge Univ. Press, Cambridge, 2001.

[16] L. Evans. *Partial Differential Equations,* Amer. Math. Soc., 1998.

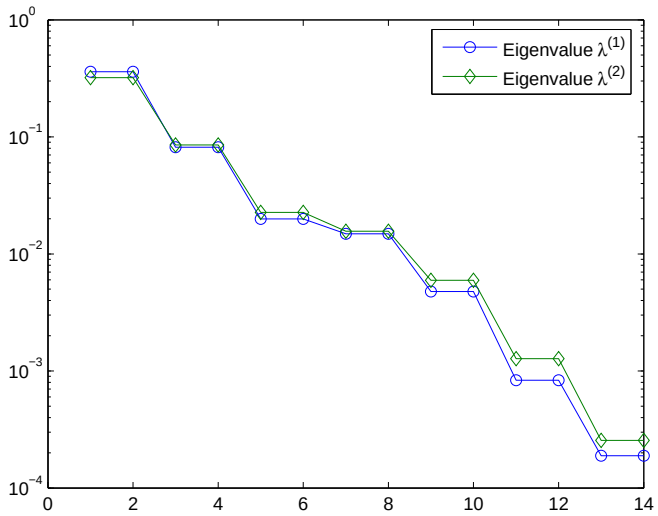

Figure 12: Ω_2: errors $\left|\lambda_{15}^{(i)} - \lambda_n^{(i)}\right|$ for the calculation of the first two eigenvalues $\lambda^{(i)}$

[17] W. Gautschi, *Orthogonal Polynomials*, Oxford University Press, Oxford, 2004.

[18] O. Hansen, K. Atkinson, and D. Chien. On the norm of the hyperinterpolation operator on the unit disk and its use for the solution of the nonlinear Poisson equation, *IMA J. Numerical Analysis* **29** (2009), 257-283, DOI: 10.1093/imanum/drm052.

[19] E.W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, Chelsea Publishing Company, New York, 1965.

[20] M. Krasnoselskii. *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, 1964.

[21] O. Ladyzhenskaya and N. Uralt'seva. *Linear and Quasilinear Elliptic Equations*, Academic Press, 1973.

[22] B. Logan. and L. Shepp. Optimal reconstruction of a function from its projections, *Duke Mathematical Journal* **42**, (1975), 645–659.

[23] T. M. MacRobert, *Spherical Harmonics*, Dover Publications, Inc., New York, 1948.

[24] S. Mikhlin. *Mathematical Physics*: *An Advanced Course*, North-Holland, 1970.

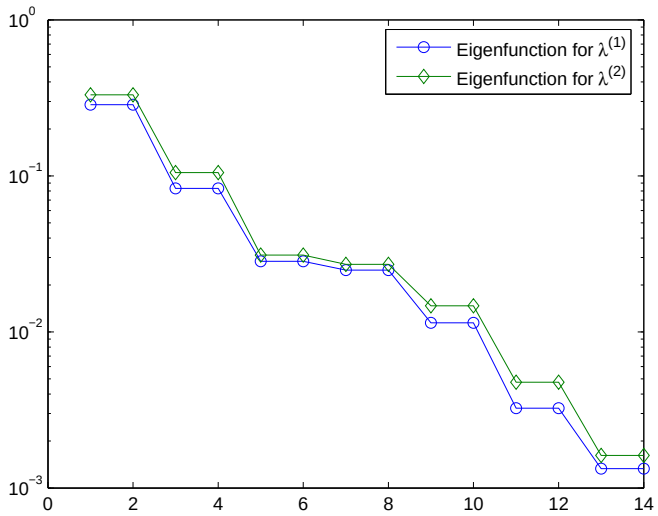

Figure 13: Ω_2: angles $\angle\left(u_n^{(i)}, u_{15}^{(i)}\right)$ between the approximate eigenfunction $u_n^{(i)}$ and our most accurate approximation $u_{15}^{(i)} \approx u^{(i)}$.

[25] J. Osborn. Spectral approximation for compact operators, *Mathematics of Computation* **29** (1975), 712-725.

[26] D. Ragozin. Constructive polynomial approximation on spheres and projective spaces, *Trans. Amer. Math. Soc.* **162** (1971), 157-170.

[27] J. Shen and T. Tang. *Spectral and High-Order Methods with Applications*, Science Press, Beijing, 2006.

[28] A. Stroud. *Approximate Calculation of Multiple Integrals,* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1971.

[29] H. Triebel. *Higher Analysis*, Hüthig Pub. Ltd., 1997.

[30] Yuan Xu. Lecture notes on orthogonal polynomials of several variables, in *Advances in the Theory of Special Functions and Orthogonal Polynomials*, Nova Science Publishers, 2004, 135-188.

[31] Yuan Xu. Analysis on the unit ball and on the simplex, *Electronic Trans. on Numer. Anal.* **25** (2006), 284-301.

[32] S. Zhang, J. Jin, *Computation of Special Functions*, John Wiley & Sons, Inc., New York, 1996.